MATH 1010E University Mathematics
Lecture Notes (week 9)
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1 Integration Rule I: u-substitution

There is a very useful way to calculate indefinite integrals using the method of substitution. This is similar in concept to the change of variables. Let us illustrate how we can do a u-substitution using an example.

Example 1.1 *Evaluate the indefinite integral*

$$\int x\sqrt{x^2+4}\,dx.$$

The integral looks difficult to evaluate at a first glance since the thing inside the square root is rather complicated. Let's try to change it to a different variable by setting

$$u = x^2 + 4,$$

which we can differentiate to get

$$\frac{du}{dx} = 2x.$$

Hence, we can define a formal expression, called the *differential*, as

$$du = 2xdx.$$

Combining all these, we can rewrite the original indefinite integral as an integral with respect to the new variable u:

$$\int x\sqrt{x^2+4}\,dx = \int \frac{1}{2}\sqrt{x^2+4}\,(2xdx) = \int \frac{\sqrt{u}}{2}\,du.$$

We can then evaluate the new integral in u,

$$\int \frac{\sqrt{u}}{2}\,du = \frac{u^{3/2}}{3} + C = \frac{(x^2+4)^{3/2}}{3} + C.$$

The question now is why it works. A general philosophy is that since *differentiation* and *integration* are basically inverse processes (which is the statements of Fundamental Theorem of Calculus), any differentiation rule should correspond to an integration rule. For example, the linearity of differentiation implies the linearity of integration:

$$\frac{d}{dx}(af(x) + bg(x)) = a\frac{d}{dx}f(x) + b\frac{d}{dx}g(x)$$

$$\Rightarrow \quad \int af(x) + bg(x)\, dx = a\int f(x)\, dx + b\int g(x)\, dx.$$

In fact, the u-substitution method in integration is based on the chain rule in differentiation. Recall the chain rule says that

$$\frac{d}{dx}f(u(x)) = f'(u(x))u'(x).$$

Therefore,

$$\int f'(u(x))u'(x)\, dx = f(u(x)) + C.$$

If we define the differential of u as

$$du := \frac{du}{dx}\, dx,$$

then we can rewrite the integral on the left hand side as an integral in u:

$$\int f'(u)\, du = \int f'(u(x))u'(x)\, dx = f(u(x)) + C.$$

This is nothing but the u-substitution method. Let us look at a few more examples.

Example 1.2 *Sometimes there is no obvious choice which substitution to make. Consider the indefinite integral*

$$\int \frac{x\, dx}{(1+x^2)^2}.$$

If we let $u = 1 + x^2$, then $du = 2xdx$. Hence the integral becomes

$$\int \frac{x\, dx}{(1+x^2)^2} = \int \frac{\frac{1}{2}du}{u^2} = \frac{1}{2}\int u^{-2}\, du = -\frac{1}{2}u^{-1} + C = -\frac{1}{2(1+x^2)} + C.$$

Instead if we let $u = (1+x^2)^2$, then $du = 4x(1+x^2)dx$. Therefore, we can express the differential

$$xdx = \frac{du}{4(1+x^2)} = \frac{du}{4\sqrt{u}}.$$

Using this in the u-substitution,

$$\int \frac{xdx}{(1+x^2)^2} = \int \frac{1}{u}\frac{du}{4\sqrt{u}} = \frac{1}{4}\int u^{-\frac{3}{2}}du = -\frac{1}{2\sqrt{u}} + C = -\frac{1}{2(1+x^2)} + C.$$

Notice that this still yields the same answer, but the calculations are more involved. Therefore, it depends on experience to make a good choice of the substitution to evaluate an indefinite integral.

Example 1.3 *Evaluate the indefinite integral* $\int \tan x\, dx$.

Recall that $\tan x = \frac{\sin x}{\cos x}$. Therefore, if we let $u = \cos x$, then the differential is $du = -\sin x\, dx$. Hence,

$$\int \tan x\, dx = \int \frac{\sin x}{\cos x} dx = \int \frac{-du}{u} = -\ln|u| + C = -\ln|\cos x| + C.$$

Example 1.4 *Sometimes we have to manipulate the integrand a bit before doing the u-substitution. For example, we consider the integral*

$$\int \sec x\, dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} dx.$$

Let $u = \sec x + \tan x$, then $du = (\sec x \tan x + \sec^2 x)\, dx$. Therefore, the integral becomes

$$\int \frac{du}{u} = \ln|u| + C = \ln|\sec x + \tan x| + C.$$

Example 1.5 *Evaluate the indefinite integral* $\int xe^{-x^2}\, dx$.

Let $u = x^2$, then the differential is $du = 2xdx$. Hence

$$\int xe^{-x^2}\, dx = \int \frac{1}{2}e^{-u}\, du = -\frac{1}{2}e^{-u} + C = -\frac{1}{2}e^{-x^2} + C.$$

Example 1.6 *When we are more experienced, sometimes we just do it without explicitly mentioning the variable u. For example,*

$$\int \frac{1}{1+x}dx = \int \frac{1}{1+x}d(1+x) = \ln|1+x| + C.$$

We can also evaluate some other integrals of rational functions.

$$\int \frac{x\, dx}{1+x} = \int \left(1 - \frac{1}{1+x}\right)\, dx = x - \ln|1+x| + C.$$

Later, we will learn a method called partial fractions *to evaluate integrals of rational functions.*

Example 1.7 *Trigonometric identities can help us evaluate some indefinite integrals. For example, using the half-angle formula,*

$$\int \sin^2 x \, dx = \int \frac{1 - \cos 2x}{2} dx = \frac{1}{2}\left(x - \frac{\sin 2x}{2}\right) + C,$$

where we have evaluated the second term by

$$\int \cos 2x \, dx = \int \frac{1}{2} \cos(2x) \, d(2x) = \frac{1}{2} \sin 2x + C.$$

Question: Evaluate the indefinite integral

$$\int \cos^2 x \, dx,$$

either using (i) the half-angle formula or (ii) the result in Example 1.7 and the identity $\cos^2 x + \sin^2 x = 1$.

2 Definite Integrals

Recall that when we discuss differentiation, there are two (related) perspectives. One is that differentiation is a way to produce a new function from an old one:

$$f(x) \xrightarrow{\frac{d}{dx}} f'(x).$$

A similar operation on the side of integration is the concept of indefinite integrals.

$$f(x) \xrightarrow{\int} \int f(x) \, dx.$$

(To be more precise, this produces a family of new functions which differed by an additive constant.)

Another way to understand differentiation, which is more geometric, is that if we fixed a point $c \in (a, b)$, where $f : (a, b) \to \mathbb{R}$ is a differentiable function, then

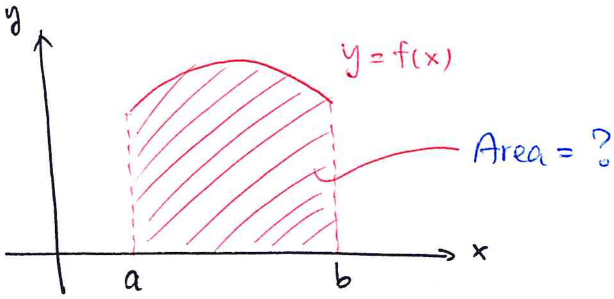
$$f(x) \xrightarrow{\frac{d}{dx}\big|_{x=c}} f'(c),$$

which is a *number* that represents the slope of the graph $y = f(x)$ at the point $(c, f(c))$. We then ask what is the corresponding concept for integration. Therefore, if $f : [a, b] \to \mathbb{R}$ is a "nice" function, then can a certain "*integration*" associate a number to this function which has some geometric meaning.

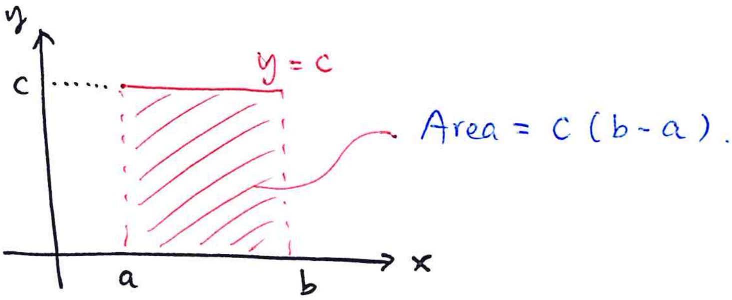
$$f(x) \xrightarrow{\int_a^b} \int_a^b f(x)\, dx.$$

It turns out that such a thing exists and is geometrically related to the following problem:

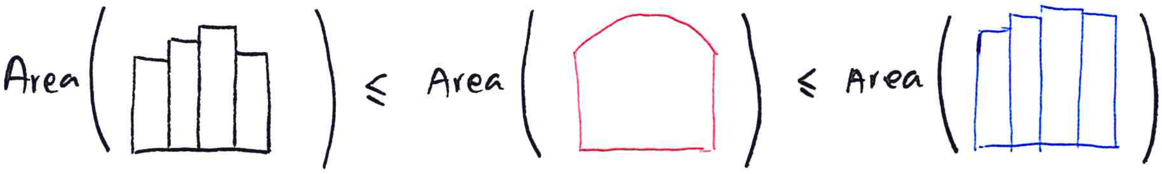
Problem: Given a continuous function $f : [a, b] \to \mathbb{R}$, how to find the area under the graph?



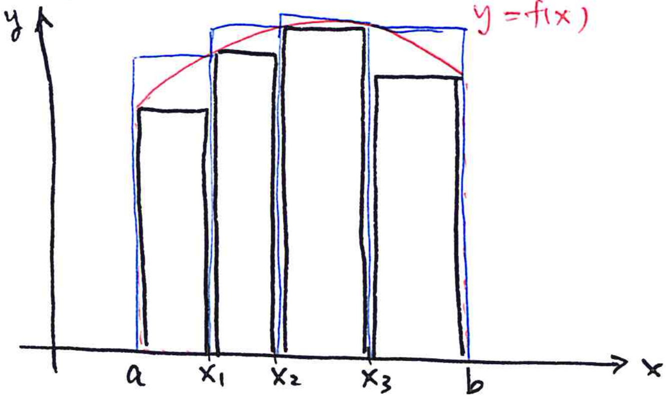
The simplest case is of course the "rectangles", which correspond to constant functions. The area is obviously given by the product of its width and its height.



For the general case, our idea is to do *approximations*! Since we know how to calculate the area of rectangles, it is then natural to use rectangles to approximate the region R under the graph $y = f(x)$. Consider first a partition of the interval $[a, b]$ into smaller subintervals. Then, over each subinterval, we draw a blue rectangle which covers the region under the graph and a black rectangle which is covered by the region under the graph. These blue rectangles hence cover the whole region under the graph, while the black rectangles are contained in the region under the graph. Therefore, we get a upper and lower estimate of the actual area under the graph by the total area of the blue and black rectangles respectively. As the partition gets finer and finer, we expect that the approximation gets better and better to the actual area we want to compute. This is precisely the concept of



Riemann integration.



3 Riemann integrability

Now, we use the general idea of approximation above to define Riemann definite integrals.

Suppose $f : [a, b] \to \mathbb{R}$ is a function.

Step 1: Take a *partition* P of the interval $[a, b]$, i.e. a sequence

$$a = x_0 < x_1 < x_2 < x_3 < \cdots < x_{n-1} < x_n = b.$$

We call the *size* of the k-th subinterval $[x_{k-1}, x_k]$, $k = 1, 2, \cdots, n$

$$\Delta x_k := x_k - x_{k-1}$$

and the *mesh* of the partition P, which measures the fineness of the partition, by

$$|P| := \max_k \Delta x_k.$$

Step 2: On each subinterval $[x_{k-1}, x_k]$, we define

$$M_k := \max_{x \in [x_{k-1}, x_k]} f(x),$$

$$m_k := \min_{x \in [x_{k-1}, x_k]} f(x).$$

(Note: We should be more careful here since the max/min may not exist in general if f is not continuous. In those cases, we have to replace max/min by sup/inf. Unlike max/min, sup/inf is the smallest upper bound/largest lower bound, which always exists for any function.)

Step 3: Approximation by rectangles. Define the upper sum and lower sum relative to the partition P by

$$\mathcal{U}(f,P) := \sum_{k=1}^{n} M_k \Delta x_k,$$

$$\mathcal{L}(f,P) := \sum_{k=1}^{n} m_k \Delta x_k.$$

Clearly, we have

$$\mathcal{L}(f,P) \leq A \leq \mathcal{U}(f,P),$$

where A is the area of the region below the graph $y = f(x)$.

Step 4: Take a limit as $|P| \to 0$, i.e. refine the partition indefinitely. If both the limit exists and equal, i.e.

$$\lim_{|P|\to 0} \mathcal{U}(f,P) = \lim_{|P|\to 0} \mathcal{L}(f,P),$$

then we say that f is *Riemann integrable on* $[a,b]$ and we denote the common limit by

$$\int_a^b f(x)\,dx,$$

which is called the *definite integral* of f on $[a,b]$.

Fact: Some functions are NOT Riemann integrable. For example, the function $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

is NOT Riemann integrable. Recall that a number x is rational if $x = p/q$ for some integers p, q.

So, it makes sense to ask which functions are Riemann integrable. It turns out that compared to differentiability, it is much easier for a function to be Riemann integrable. In fact we have the following theorem.

Theorem 3.1 *Any function $f : [a,b] \to \mathbb{R}$ which is bounded and continuous except only at finitely many points is Riemann integrable. In particular, any continuous function $f : [a,b] \to \mathbb{R}$ is Riemann integrable.*

We will skip the proof of the theorem. The basic idea is that the jump discontinuity causes error with zero area. For interested students, please consult any textbook on mathematical analysis.

4 Definite integrals as Riemann Sums

From the last theorem in the previous section, we know that continuous functions are Riemann integrable. In fact, we know more: not only are they integrable, but the definite integrable can also be calculated by a *Riemann sum*.

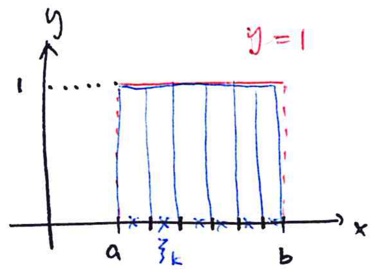
Theorem 4.1 *If $f : [a,b] \to \mathbb{R}$ is a continuous function, then it is Riemann integrable and its definite integral can be computed by the Riemann sum:*

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(\xi_k)\frac{b-a}{n},$$

where $\xi_k \in [x_{k-1}, x_k]$ is ANY point in the k-th subinterval of the uniform partition P:

$$a = x_0 < x_1 := x_0 + \frac{b-a}{n} < \cdots < x_k := x_0 + k\left(\frac{b-a}{n}\right) < \cdots < x_n = b.$$

Hence, the theorem says that we could just use uniform partitions for approximations and we can take any sample points ξ_k in each subinterval. Let's look at some examples.



Example 4.2 *Use the method of Riemann sum to show that*

$$\int_a^b 1\,dx = b - a.$$

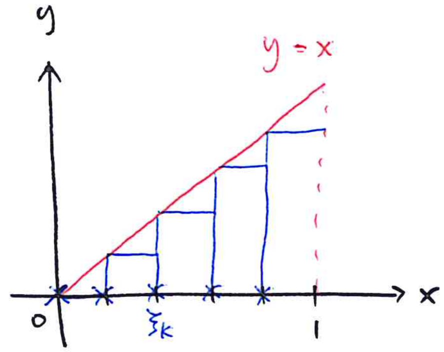
Since $f(x) = 1$ is a constant function, so

$$\int_a^b 1\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(\xi_k)\frac{b-a}{n} = \lim_{n\to\infty} \sum_{k=1}^{n} 1 \cdot \frac{b-a}{n} = \lim_{n\to\infty} (b-a) = b - a.$$

Example 4.3 *Use the method of Riemann sum to show that*

$$\int_0^1 x\,dx = \frac{1}{2}.$$

Let $f(x) = x$. If we take $\xi_k = \frac{k-1}{n} \in [\frac{k-1}{n}, \frac{k}{n}]$, then



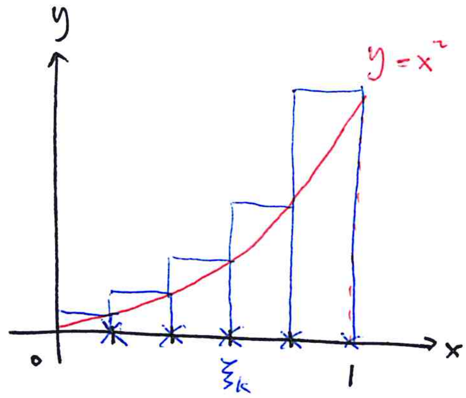
$$\begin{aligned}\int_0^1 x\,dx &= \lim_{n\to\infty} \sum_{k=1}^{n} f\left(\frac{k-1}{n}\right)\frac{1}{n} \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{k-1}{n}\cdot\frac{1}{n} \\ &= \lim_{n\to\infty} \frac{1}{n^2}\sum_{k=1}^{n-1} k \\ &= \lim_{n\to\infty} \frac{n(n-1)}{2n^2} = \frac{1}{2}.\end{aligned}$$

Note that we have used the fact that

$$\sum_{k=1}^{n-1} k = \frac{n(n-1)}{2}.$$

Example 4.4 *Use the method of Riemann sum to show that*

$$\int_0^1 x^2\,dx = \frac{1}{3}.$$



Let $f(x) = x^2$. If we take $\xi_k = \frac{k}{n} \in [\frac{k-1}{n}, \frac{k}{n}]$, then

$$\begin{aligned}\int_0^1 x^2\,dx &= \lim_{n\to\infty} \sum_{k=1}^{n} f\left(\frac{k}{n}\right)\frac{1}{n} \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^2}{n^2}\cdot\frac{1}{n} \\ &= \lim_{n\to\infty} \frac{1}{n^3}\sum_{k=1}^{n} k^2 \\ &= \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6n^3} = \frac{1}{3}.\end{aligned}$$

Note that we have used the fact that

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

Question: Try to evaluate the Riemann sum by taking different ξ_k's.

Example 4.5 *Use the method of Riemann sum to show that*

$$\int_0^1 e^x \, dx = e - 1.$$

Let $f(x) = e^x$. If we take $\xi_k = \frac{k}{n} \in [\frac{k-1}{n}, \frac{k}{n}]$, then

$$\begin{aligned}
\int_0^1 e^x \, dx &= \lim_{n\to\infty} \sum_{k=1}^{n} e^{\frac{k}{n}} \frac{1}{n} \\
&= \lim_{n\to\infty} \frac{1}{n}(e^{\frac{1}{n}} + e^{\frac{2}{n}} + e^{\frac{3}{n}} + \cdots + e^{\frac{n}{n}}) \\
&= \lim_{n\to\infty} \frac{1}{n} \frac{e^{\frac{1}{n}}(1 - e^{\frac{n}{n}})}{1 - e^{\frac{1}{n}}} \\
&= (1-e) \lim_{n\to\infty} \frac{\frac{1}{n}e^{\frac{1}{n}}}{1 - e^{\frac{1}{n}}} \\
&= (1-e) \lim_{x\to 0} \frac{xe^x}{1-e^x} = e - 1.
\end{aligned}$$

We have used the following formula for a geometric series

$$a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(1-r^n)}{1-r},$$

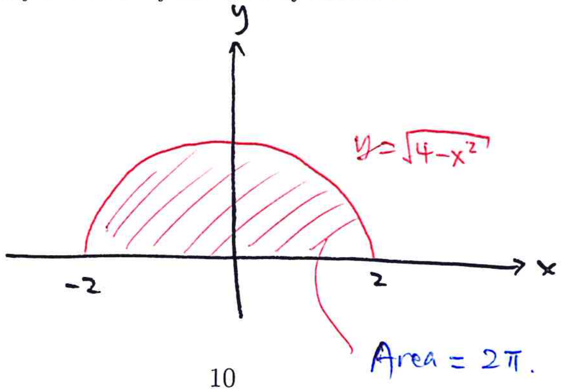
when $r \neq 1$. The last limit can be evaluated by the L'Hospital's Rule.

Sometimes we can use the geometric meaning of definite integral to help us in the calculation.

Example 4.6 *We know that*

$$\int_{-2}^{2} \sqrt{4 - x^2} \, dx = \frac{1}{2}\pi(2)^2 = 2\pi,$$

since it is simply half the area of the disk of radius 2.



Proposition 4.7 (Properties of definite integrals) *The following properties hold for definite integrals:*

(1) $\displaystyle\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$

(2) $\displaystyle\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$ *for any constant* k.

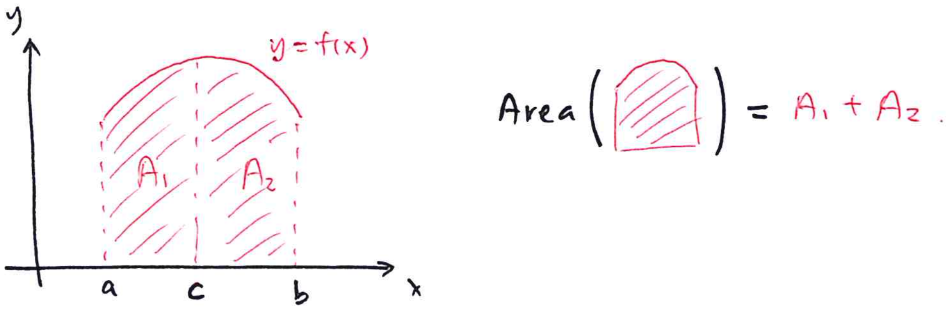
(3) If $a < b$, *then we have (define)*

$$\int_b^a f(x)\,dx := -\int_a^b f(x)\,dx.$$

(4) For any $c \in (a, b)$, *we have*

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

In fact, the same holds even for any $c > b$ *or* $c < a$, *as long as* f *is continuously defined everywhere.*



We will finish this section by a special example that uses symmetry to evaluate a definite integral.

Example 4.8 *Show that*

$$\int_0^{\frac{\pi}{2}} \sin^2 x\,dx = \frac{\pi}{4}.$$

Recall that we have the symmetry $\cos(\frac{\pi}{2} - x) = \sin x$ for any x. Hence,

$$\int_0^{\frac{\pi}{2}} \sin^2 x\,dx = \int_0^{\frac{\pi}{2}} \cos^2\left(\frac{\pi}{2} - x\right)\,dx.$$

Let $u = \pi/2 - x$, then $du = -dx$. Moreover, when $x = 0$, $u = \pi/2$ and when $x = \pi/2$, $u = 0$. Therefore,

$$\int_0^{\frac{\pi}{2}} \cos^2\left(\frac{\pi}{2} - x\right)\, dx = \int_{\frac{\pi}{2}}^0 \cos^2 u\ (-du) = \int_0^{\frac{\pi}{2}} \cos^2 u\ du = \int_0^{\frac{\pi}{2}} \cos^2 x\ dx.$$

In other words, we have

$$\int_0^{\frac{\pi}{2}} \sin^2 x\ dx = \int_0^{\frac{\pi}{2}} \cos^2 x\ dx.$$

Moreover, since $\sin^2 x + \cos^2 x = 1$, integrating on both sides we get

$$\int_0^{\frac{\pi}{2}} \sin^2 x\ dx + \int_0^{\frac{\pi}{2}} \cos^2 x\ dx = \int_0^{\frac{\pi}{2}} 1\ dx = \frac{\pi}{2}.$$

Since the two integrals on the left are equal, we conclude that

$$\int_0^{\frac{\pi}{2}} \sin^2 x\ dx = \int_0^{\frac{\pi}{2}} \cos^2 x\ dx = \frac{\pi}{4}.$$